13.7: 4, 10, 14, 16, 18

4. The boundary curve C is the circle $x^2 + z^2 = 9$, $y = 3$ with vector equation $\mathbf{r}(t) = 3\sin t\,\mathbf{i} + 3\,\mathbf{j} + 3\cos t\,\mathbf{k}$, $0 \le t \le 2\pi$ which gives the positive orientation. Then $\mathbf{F}(\mathbf{r}(t)) = 729\sin^2 t\cos t\,\mathbf{i} + \sin(27\sin t\cos t)\,\mathbf{j} + 27\sin t\cos t\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = 2187\sin^2 t\cos^2 t - 81\sin^2 t\cos t$. Thus

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt \\ &= \int_0^{2\pi}(2187\sin^2 t\cos^2 t - 81\sin^2 t\cos t)\,dt = \int_0^{2\pi}\left(2187\left(\tfrac{1}{2}\sin 2t\right)^2 - 81\sin^2 t\cos t\right)dt \\ &= \left[\tfrac{2187}{4}\left(\tfrac{1}{2}t - \tfrac{1}{8}\sin 4t\right) - 81\cdot\tfrac{1}{3}\sin^3 t\right]_0^{2\pi} = \tfrac{2187}{4}(\pi) - 0 = \tfrac{2187}{4}\pi\end{aligned}$$

10. The curve of intersection is an ellipse in the plane $z = 5 - x$. $\operatorname{curl}\mathbf{F} = \mathbf{i} - x\,\mathbf{k}$ and we take the surface S to be the planar region enclosed by C with upward orientation, so

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint\limits_{x^2+y^2\le 9}[-1(-1) - 0 + (-x)]\,dA = \int_0^{2\pi}\int_0^3 (1 - r\cos\theta)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^3 \left(r - r^2\cos\theta\right)dr\,d\theta = \int_0^{2\pi}\left(\tfrac{9}{2} - 9\cos\theta\right)d\theta = \left[\tfrac{9}{2}\theta - 9\sin\theta\right]_0^{2\pi} = 9\pi\end{aligned}$$

14. The plane intersects the coordinate axes at $x = 1$, $y = z = 2$ so the boundary curve C consists of the three line segments C_1: $\mathbf{r}_1(t) = (1-t)\,\mathbf{i} + 2t\,\mathbf{j}$, $0 \le t \le 1$, C_2: $\mathbf{r}_2(t) = (2-2t)\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \le t \le 1$, C_3: $\mathbf{r}_3(t) = t\,\mathbf{i} + (2-2t)\,\mathbf{k}$, $0 \le t \le 1$. Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 [(1-t)\,\mathbf{i} + 2t\,\mathbf{j}]\cdot(-\mathbf{i} + 2\,\mathbf{j})\,dt + \int_0^1 [(2-2t)\,\mathbf{j}]\cdot(-2\,\mathbf{j} + 2\,\mathbf{k})\,dt + \int_0^1 (t\,\mathbf{i})\cdot(\mathbf{i} - 2\,\mathbf{k})\,dt \\ &= \int_0^1 (5t-1)\,dt + \int_0^1 (4t-4)\,dt + \int_0^1 t\,dt = \tfrac{3}{2} - 2 + \tfrac{1}{2} = 0\end{aligned}$$

Now $\operatorname{curl}\mathbf{F} = xz\,\mathbf{i} - yz\,\mathbf{j}$, so by Equation 13.6.10 with $z = g(x,y) = 2 - 2x - y$ we have

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D [-x(2-2x-y)(-2) + y(2-2x-y)(-1)]\,dA \\ &= \int_0^1\int_0^{2-2x}(4x - 4x^2 - 2y + y^2)\,dy\,dx \\ &= \int_0^1 \left[4x(2-2x) - 4x^2(2-2x) - (2-2x)^2 + \tfrac{1}{3}(2-2x)^3\right]dx \\ &= \int_0^1 \left(\tfrac{16}{3}x^3 - 12x^2 + 8x - \tfrac{4}{3}\right)dx = \left[\tfrac{4}{3}x^4 - 4x^3 + 4x^2 - \tfrac{4}{3}x\right]_0^1 = 0\end{aligned}$$

16. The components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$, and both the curve C and the surface S meet the requirements of Stokes' Theorem. If there is a vector field $\mathbf{G}$ where $\mathbf{F} = \operatorname{curl}\mathbf{G}$, then Stokes' Theorem says $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \operatorname{curl}\mathbf{G}\cdot d\mathbf{S}$ depends only on the values of $\mathbf{G}$ on C, and hence is independent of the choice of S. By Theorem 13.5.11, $\operatorname{div}\operatorname{curl}\mathbf{G} = 0$, so $\operatorname{div}\mathbf{F} = 0 \iff (3ax^2 - 3z^2) + (x^2 + 3by^2) + (3cz^2) = 0 \iff$ $(3a+1)x^2 + 3by^2 + (3c-3)z^2 = 0 \iff a = -\tfrac{1}{3}$, $b = 0$, $c = 1$.

18. $\int_C (y+\sin x)\,dx + (z^2+\cos y)\,dy + x^3\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y+\sin x)\,\mathbf{i} + (z^2+\cos y)\,\mathbf{j} + x^3\,\mathbf{k} \quad\Rightarrow$ $\operatorname{curl}\mathbf{F} = -2z\,\mathbf{i} - 3x^2\,\mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2\sin t\cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto the xy-plane is the unit disk D $(x^2+y^2\le 1)$. C is traversed clockwise (when viewed from above) so S is oriented downward. Using Equation 13.6.10 with $g(x,y) = 2xy$, $P = -2(2xy) = -4xy$, $Q = -3x^2$, $R = -1$, we have

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= -\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-(-4xy)(2y) - (-3x^2)(2x) - 1\right] dA \\ &= -\iint_D (8xy^2 + 6x^3 - 1)\,dA = -\int_0^{2\pi}\int_0^1 (8r^3\cos\theta\sin^2\theta + 6r^3\cos^3\theta - 1)\,r\,dr\,d\theta \\ &= -\int_0^{2\pi}\left(\tfrac{8}{5}\cos\theta\sin^2\theta + \tfrac{6}{5}\cos^3\theta - \tfrac{1}{2}\right) r\,dr\,d\theta = -\left[\tfrac{8}{15}\sin^3\theta + \tfrac{6}{5}\left(\sin\theta - \tfrac{1}{3}\sin^3\theta\right) - \tfrac{1}{2}\theta\right]_0^{2\pi} = \pi\end{aligned}$$